MVE030

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Theorems marked with * are included on the theory list. Note that unlike some other courses, all statements to be proved are given in their entirety. Additional theorems, definitions and remarks are included to provide suitable background or techniques for difficult problems. This material is subject to change.

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List of Theorems

Definition (Fourier coefficients)**.** The Fourier coefficients c_n of a function f on $[-\pi, \pi]$ are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \tag{1}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \tag{2}$$

Theorem (Bessel's inequality)**.** *Assume* $f \in \mathcal{L}^2([a,b])$*. Then*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx \tag{3}$$

Corollary. *Since the series in the equality converges, we necessarily have*

$$\lim_{n\to\pm\infty} c_n = 0 \quad \textit{and} \quad \lim_{n\to\pm\infty} a_n = 0, \quad \lim_{n\to\pm\infty} b_n = 0$$

Remark (Dirichlet kernel)*.* The N*th Dirichlet kernel*

$$D_N(t) := \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int} = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{-int} = \frac{1}{2\pi} \left(1 + \sum_{n=1}^{N} 2\cos(nt)\right) \tag{4}$$

has two notable properties:

$$\int_0^{\pi} D_N(t)\, dt = \int_{-\pi}^{0} D_N(t)\, dt = \frac{1}{2} \tag{5}$$

since D_N is even, and using geometric sums

$$D_N(t) = \frac{e^{-iNt}}{2\pi} \sum_{n=0}^{2N} e^{iNt} = \frac{e^{-iNt}}{2\pi} \frac{1 - e^{i(2N+1)t}}{1 - e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})} \tag{6}$$

Theorem* (2.1, pointwise convergence of Fourier series)**.** *Assume that f is piecewise* C^1 *on* $[-\pi,\pi]$ *and* f *is defined (extended) to be* 2π*-periodic on* $\mathbb{R}$*. Then*

$$S_N(x) := \sum_{n=-N}^{N} c_n e^{inx} \tag{7}$$

satisfies

$$\lim_{N\to\infty} S_N(x) = \frac{f(x_+) + f(x_-)}{2} \;\forall x \in \mathbb{R} \tag{8}$$

where $f(x_+) := \lim_{t\to x^+} f(t)$, $f(x_-) := \lim_{t\to x^-} f(t)$.

Thus, when f *is continuous at* x*, then* $f(x_+) = f(x_-) = f(x)$*, so the Fourier series converges to f(x).*

Proof. Fix an arbitrary $x \in \mathbb{R}$. Our goal is to prove

$$\lim_{N\to\infty} \left| S_N(x) - \frac{f(x_+) + f(x_-)}{2} \right| = 0 \tag{9}$$

We insert the definition from (1).

$$S_N(x) = \sum_{|n|\le N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right) e^{inx} = \sum_{|n|\le N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{in(x-y)}\, dy \tag{10}$$

Let $t := y - x$. Then $y = t + x \implies dy = dt$. We may shift the bounds of integration because the integral is over a whole period.

$$S_N(x) = \sum_{|n|\le N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x) e^{-int}\, dt = \sum_{|n|\le N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x) e^{-int}\, dt \tag{11}$$

We rewrite using the definition of D_N.

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) \left(\sum_{|n|\le N} \frac{1}{2\pi} e^{-int} \right) dt = \int_{-\pi}^{\pi} f(t+x) D_N(t)\, dt \tag{12}$$

Returning to our goal (9), we make use of this expression and properties of D_N.

$$\left| \int_{-\pi}^{\pi} f(t+x) D_N(t)\, dt - \frac{f(x_+) + f(x_-)}{2} \right| \tag{13}$$

$$\overset{(5)}{=} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t)\, dt - \int_{-\pi}^{0} f(x_-) D_N(t)\, dt - \int_{0}^{\pi} f(x_+) D_N(t)\, dt \right| \tag{14}$$

$$= \left| \int_{-\pi}^{0} (f(t+x) - f(x_-)) D_N(t)\, dt + \int_{0}^{\pi} (f(t+x) - f(x_+)) D_N(t)\, dt \right| \tag{15}$$

$$\overset{(6)}{=} \left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})} (f(t+x) - f(x_-))\, dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1 - e^{it})} (f(t+x) - f(x_+))\, dt \right| \tag{16}$$

Let $g(t)$ be as follows:

$$g(t) := \begin{cases} \frac{f(t+x) - f(x_-)}{1 - e^{it}} & -\pi \le t < 0 \\ \frac{f(t+x) - f(x_+)}{1 - e^{it}} & 0 < t \le \pi \end{cases} \tag{17}$$

Then $g(t)$ is piecewise C^1, since

$$\frac{f(t+x) - f(x_-)}{1 - e^{it}} = \frac{t(f(t+x) - f(x_-))}{t(1 - e^{it})} \underset{t\to 0^-}{\to} \frac{f'(x_-)}{-ie^{i\cdot 0}} = \frac{f'(x_-)}{-i} \tag{18}$$

and similarly for the right-hand limit. By Weierstraß' theorem (on each part), $g(t)$ is bounded on $[-\pi,\pi]$. We can now condense the long expression (16) into

$$= \left| \int_{-\pi}^{\pi} \frac{e^{-iNt}}{2\pi} g(t)\, dt - \int_{-\pi}^{\pi} \frac{e^{i(N+1)t}}{2\pi} g(t)\, dt \right| \tag{19}$$

However, these terms are precisely the Nth and $(-N-1)$th Fourier coefficients for g, respectively. By corollary to Bessel's inequality, these tend to 0 as $N \to \infty$, which completes the proof. □

Theorem* (2.2, differentiation of Fourier series). *Assume that f is piecewise C^1 on $(-\pi,\pi)$, 2π-periodic and continuous on $\mathbb{R}$. Then the Fourier coefficients of f: a_n, b_n, c_n and those of f': a'_n, b'_n, c'_n satisfy*

$$a'_n = nb_n, \qquad b'_n = -na_n, \qquad c'_n = inc_n \tag{20}$$

Proof.

$$c'_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx \tag{21}$$

$$\overset{\text{I.P.}}{=} \frac{1}{2\pi}\left(\cancelto{0}{e^{-inx}f(x)\big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi}(-in)e^{-inx}f(x)\,dx\right) \tag{22}$$

$$= \frac{in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = inc_n \tag{23}$$

where the cancellation is by 2π-periodicity, and the other expressions follow from the relations $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$. □

Theorem (integration of Fourier series). *Assume that f is piecewise continuous and 2π-periodic. Define $F(x) := \int_0^x f(t)\,dt$. As long as $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$, then*

$$F(x) = C_0 + \sum_{n\neq 0}\frac{c_n}{in}e^{inx} \tag{24}$$

where $C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\,dx = 0$

Remark (Series on arbitrary intervals). One may generalize the coefficent formulas to some interval $[a-l, a+l]$, $a\in\mathbb{R}$, $l>0$ (the length is then $2l$).

$$c_n = \frac{1}{2l}\int_{a-l}^{a+l} f(x)e^{-in(x-a)\pi/l}\,dx \tag{25}$$

$$a_n = \frac{1}{l}\int_{a-l}^{a+l} f(x)\cos\left((n(x-a)\frac{\pi}{l}\right)\,dx \quad b_n = \frac{1}{l}\int_{a-l}^{a+l} f(x)\sin\left(n(x-a)\frac{\pi}{l}\right)\,dx \tag{26}$$

The corresponding series are

$$\sum_{n\in\mathbb{Z}} c_n e^{in(x-a)\pi/l} \tag{27}$$

$$\frac{a_0}{2} + \sum_{n\geq 1}\left[a_n\cos\left((n(x-a)\frac{\pi}{l}\right) + b_n\sin\left((n(x-a)\frac{\pi}{l}\right)\right] \tag{28}$$

Theorem (Rate of convergence)**.** *Assume f is 2π-periodic, f is C^{k-1} and $f^{(k-1)}$ is pw C^1 $\implies$ f is pw C^k. Then the Fourier coefficients $C_n = \{a_n, b_n, c_n\}$ satisfy*

$$\sum_{n\in\mathbb{Z}} |n^k C_n|^2 < \infty \tag{29}$$

Proposition (properties of the inner product)**.** *For $u,v,w \in H$,*

(i) $\langle u,v\rangle = \overline{\langle v,u\rangle}$

(ii) $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

(iii) $\langle au,v\rangle = a\langle u,v\rangle$, $\forall a \in \mathbb{C}$

(iv) $\|u\|^2 = \langle u,u\rangle \geq 0,\ \|u\| = 0 \iff u = 0$

(v) The inner product is continuous: If $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ are in H, and $\lim_{n\to\infty} u_n = u \in H$, $\lim_{n\to\infty} v_n = v \in H$, then $\lim_{n\to\infty}\langle u_n,v_n\rangle = \langle u,v\rangle$.

Proposition (Hilbert space propositions)**.** *For any Hilbert space H*

(i) Cauchy-Schwarz: $u,v \in H : |\langle u,v\rangle| \leq \|u\|\,\|v\|$

(ii) Triangle ineq.: $u,v \in H : \|u+v\| \leq \|u\| + \|v\|$

(iii) n-dim Pythagorean Theorem: If $\{u_k\}_{k=1}^n \in H$ are pairwise orthogonal then $\|u_1 + \cdots + u_n\|^2 = \|u_1\|^2 + \cdots + \|u_n\|^2$
In fact, n may tend to infinity as a consequence of the continuity of the inner product.

Theorem (Bessel's inequality redux)**.** *Given that $\{\phi_n\}_{n\geq 1}$ is an orthonormal set in a Hilbert space H, then*

$$\forall f \in H : g := \sum_{n\geq 1} \langle f,\phi_n\rangle\,\phi_n \in H \tag{30}$$

and moreover, $\|g\|^2 \leq \|f\|^2$.

Theorem* (3.4, ONB conditions)**.** *If $f \in H$, the following are equivalent:*

(i) $\langle f,\phi_n\rangle = 0\ \forall n \implies f = 0$

(ii) $\sum_{n\geq 1}\langle f,\phi_n\rangle\,\phi_n = f$

(iii) $\|f\|^2 = \sum_{n\geq 1} |\langle f,\phi_n\rangle|^2$ *(Parseval's equation).*

Proof. $(i \Rightarrow ii)$: By Bessel's ineq. $g := \sum_{n\geq 1} \langle f,\phi_n \rangle \phi_n \in H$. Consider

$$\langle f - g, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle = \langle f, \phi_n \rangle - \left\langle \sum_{m\geq 1} \langle f,\phi_m \rangle \phi_m, \phi_n \right\rangle \tag{31}$$

$$\overset{*}{=} \langle f, \phi_n \rangle - \sum_{m\geq 1} \langle f,\phi_m \rangle \underbrace{\langle \phi_m, \phi_n \rangle}_{=\delta_{mn}} = \langle f,\phi_n \rangle - \langle f,\phi_n \rangle = 0 \tag{32}$$

where it should be noted that the marked eq. uses both linearity and continuity in the inner product. By (i), $\langle f - g, \phi_n \rangle = 0 \implies f - g = 0 \implies f = g$ i.e. (ii).

$(ii \Rightarrow iii)$: By (ii), $f = \sum_{n\geq 1} \langle f,\phi_n \rangle \phi_n$. Infinite P.T. implies

$$\|f\|^2 = \sum_{n\geq 1} \|\langle f,\phi_n \rangle \phi_n\|^2 = \sum_{n\geq 1} |\langle f,\phi_n \rangle|^2 \|\phi_n\|^2 = \sum_{n\geq 1} |\langle f,\phi_n \rangle|^2 \tag{33}$$

i.e. (iii).

$(iii \Rightarrow i)$: If $\langle f,\phi_n \rangle = 0 \ \forall n$, and by (iii) $\|f\|^2 = \sum_{n\geq 1} |\langle f,\phi_n \rangle|^2$ then $\|f\|^2 = 0$ and, by definition of $\|f\|^2$, $f = 0$ i.e. (i). This completes the cycle of implications and therefore the proof of equivalence. □

Theorem* (3.8, best approximation)**.** *Let H be a Hilbert space and $\{\phi_n\}_{n\geq 1}$ be an ONS in H. Then $\forall \sum_{n\geq 1} c_n\phi_n \in H, c_n \in \mathbb{C} \ \forall n \geq 1$ we have*

$$\left\| f - \sum_{n\geq 1} \langle f,\phi_n \rangle \phi_n \right\| \leq \left\| f - \sum_{n\geq 1} c_n\phi_n \right\| \tag{34}$$

with equality iff $c_n = \langle f,\phi_n \rangle \ \forall n \geq 1$.

Proof. Let $\hat{f}_n := \langle f,\phi_n \rangle$. These are the Fourier coeffs. of f w.r.t. $\{\phi_n\}_{n\geq 1}$, and $\sum_{n\geq 1} \hat{f}_n\phi_n$ is the corresponding series. Let $g := \sum_{n\geq 1} \hat{f}_n\phi_n$, $\varphi := \sum_{n\geq 1} c_n\phi_n$.

$$\|f - \varphi\|^2 = \|f - g + g - \varphi\|^2 = \|f - g\|^2 + 2\,\mathrm{Re}\,\langle f - g, g - \varphi \rangle + \|g - \varphi\|^2 \tag{35}$$

Consider the centre term:

$$\langle f - g, g - \varphi \rangle = \langle f, g - \varphi \rangle - \langle g, g - \varphi \rangle = \langle f,g \rangle - \langle f,\varphi \rangle - \langle g,g \rangle + \langle g,\varphi \rangle \tag{36}$$

$$= \left\langle f, \sum_{n\geq 1} \hat{f}_n\phi_n \right\rangle - \left\langle f, \sum_{n\geq 1} c_n\phi_n \right\rangle - \left\| \sum_{n\geq 1} \hat{f}_n\phi_n \right\|^2 + \left\langle \sum_{m\geq 1} \hat{f}_m\phi_m, \sum_{k\geq 1} c_k\phi_k \right\rangle \tag{37}$$

$$= \cancel{\sum_{n\geq 1} \hat{f}_n\overline{\hat{f}_n}} - \cancel{\sum_{n\geq 1} \hat{f}_n\overline{c_n}} - \cancel{\sum_{n\geq 1} |\hat{f}_n|^2} + \cancel{\sum_{m,k\geq 1} \hat{f}_m\overline{c_k} \underbrace{\langle \phi_m,\phi_k \rangle}_{=\delta_{mk}}} = 0 \tag{38}$$

Thus $\|f - \varphi\|^2 \geq \|f - g\|^2$ with equality iff $\|g - \varphi\|^2 = 0$.

$$\|g-\varphi\|^2 = \left\| \sum_{n\geq 1} (\hat{f}_n - c_n)\phi_n \right\|^2 \stackrel{(P.T.)}{=} \sum_{n\geq 1} \left\| (\hat{f}_n - c_n)\phi_n \right\|^2 = \sum_{n\geq 1} |\hat{f}_n - c_n|^2 \quad (39)$$

This is 0 iff $|\hat{f}_n - c_n|^2 = 0\ \forall n \iff \hat{f}_n - c_n = 0\ \forall n \iff \hat{f}_n = c_n\ \forall n.$ □

Corollary. *Assume that $\{\phi_n\}$ is an orthogonal set. Then the best approx. to $f \in H$ which is of the form $\sum c_n\phi_n$ is given by taking*

$$c_n = \frac{\langle f,\phi_n \rangle}{\|\phi_n\|^2} = \frac{\hat{f}_n}{\|\phi_n\|^2} \quad (40)$$

Theorem* (3.9ab, SLP facts)**.** *Let f and g be eigenfunctions for a regular SLP on $[a,b]$ with $w(x) > 0$. Then if λ, μ are the eigenvalues for f, g respectively*

1. *$\lambda, \mu \in \mathbb{R}$, and*
2. *$\lambda \neq \mu \implies \langle f,g \rangle_w = 0$.*

Proof. Recall that a regular SLP has self-adjoint boundary conditions, which guarantees that for all u,v satisfying the b.c. we have $\langle Lu,v \rangle = \langle u,Lv \rangle$. In addition, the eigenfunctions are solutions to the problem, i.e. $L(f) + \lambda f w = 0$ and $L(g) + \mu g w = 0$ are true.

Using the self-adjoint condition, we obtain two expresssions for $\langle Lf,f \rangle$:

$$\langle Lf,f \rangle = \langle -\lambda f w, f \rangle = -\lambda \int_a^b f(x)w(x)\overline{f(x)}\,dx = -\lambda \|f\|_w \quad (41)$$

$$= \langle f,Lf \rangle = \langle f, -\lambda f w \rangle = -\overline{\lambda} \langle f, fw \rangle = -\overline{\lambda} \|f\|_w \quad (42)$$

$$\implies \lambda \|f\|_w = \overline{\lambda} \|f\|_w \implies \lambda = \overline{\lambda} \implies \lambda \in \mathbb{R} \quad (43)$$

since an eigenfunction must be non-zero and therefore $\|f\|_w > 0$.

We apply similar reasoning to $\langle Lf,g \rangle$:

$$\langle Lf,g \rangle = \langle -\lambda f w, g \rangle = -\lambda \int_a^b f(x)w(x)\overline{g(x)}\,dx = -\lambda \langle f,g \rangle_w \quad (44)$$

$$= \langle f,Lg \rangle = \langle f, -\mu g w \rangle = -\overline{\mu} \langle f, gw \rangle = -\mu \langle f,g \rangle_w \quad (45)$$

$$\implies \lambda \langle f,g \rangle_w = \mu \langle f,g \rangle_w \stackrel{\lambda \neq \mu}{\implies} \langle f,g \rangle_w = 0 \quad (46)$$

since we proved above that $\lambda,\mu \in \mathbb{R}$ and it is known that $w(x)$ is real. If the inner product were not zero a contradiction would occur given that $\lambda \neq \mu$. □

Definition (Fourier transform)**.** Let $f \in \mathcal{L}^1(\mathbb{R})$, $\xi \in \mathbb{R}$.

$$\mathcal{F}(f)(\xi) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}} e^{ix\xi} f(x)\, dx \tag{47}$$

Definition (Convolution)**.** Assuming the integral converges

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\, dy \tag{48}$$

Proposition (Properties of the convolution)**.** *Let $f,g,h \in \mathcal{L}^2(\mathbb{R})$.*

(i) $|f * g| \leq \|f\|_{\mathcal{L}^2} \|g\|_{\mathcal{L}^2}$

(ii) $f * (ag + bh) = a(f * g) + b(f * h)$, $\forall a,b \in \mathbb{C}$

(iii) $f * g = g * f$

(iv) $f * (g * h) = (f * g) * h$

Proposition (Properties of the FT)**.** *Assume everything is well-defined.*

(i) $\mathcal{F}(f(x-a))(\xi) = e^{ia}\widehat{f}(\xi)$

(ii) $\mathcal{F}(f')(\xi) = i\xi\widehat{f}(\xi)$

(iii) $\mathcal{F}(xf(x))(\xi) = i\mathcal{F}(f')(\xi)$

(iv) $\mathcal{F}(f * g)(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

Proposition* (Fourier Inversion Theorem)**.** *For any $f \in \mathcal{L}^2(\mathbb{R})$*

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi)\, d\xi \tag{49}$$

Theorem* (Plancharel's Theorem)**.** *For any $f,g \in \mathcal{L}^2$*

$$\langle \widehat{f}, \widehat{g} \rangle = 2\pi \langle f,g \rangle \tag{50}$$

and thus $\|\widehat{f}\|^2_{\mathcal{L}^2} = 2\pi \|f\|^2_{\mathcal{L}^2}$

Proof.

$$2\pi \langle f,g \rangle = 2\pi \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx \overset{\text{(IF)}}{=} \frac{2\pi}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi)\, d\xi \right) \overline{g(x)}\, dx \tag{51}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{ix\xi} \overline{g(x)}\, dx \right) \widehat{f}(\xi)\, d\xi = \int_{\mathbb{R}} \overline{\left(\int_{\mathbb{R}} e^{-ix\xi} g(x)\, dx \right)} \widehat{f}(\xi)\, d\xi \tag{52}$$

$$= \int_{\mathbb{R}} \overline{\widehat{g}(\xi)} \widehat{f}(\xi)\, d\xi = \langle \widehat{f}, \widehat{g} \rangle \tag{53}$$

and the second statement follows directly by taking $f = g$. □

Theorem* (7.3, Convolution approximation)**.** *Let $g \in \mathcal{L}^1(\mathbb{R})$ with $\int_{\mathbb{R}} g(x)dx = 1$. Define $\alpha := \int_{-\infty}^0 g(x)dx$, $\beta := \int_0^\infty g(x)dx$, and $g_\varepsilon(x) := g(x/\varepsilon)/\varepsilon$. Suppose f is piecewise continuous on $\mathbb{R}$ and either (1) f is bounded, or (2) g vanishes outside a bounded interval. Then*

$$\lim_{\varepsilon\to 0}(f * g_\varepsilon)(x) = \alpha f(x_+) + \beta f(x_-) \;\forall x \in \mathbb{R} \tag{54}$$

where $f(x_+), f(x_-)$ are shorthands for the left and right hand limits as seen above. If f is continuous, then $\lim_{\varepsilon\to 0}(f * g_\varepsilon)(x) = f(x)$.

Proof. By definition of convolution, the goal is to prove

$$\lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}} f(x-y)g_\varepsilon(y)\,dy - \alpha f(x_+) - \beta f(x_-)\right] \overset{?}{=} 0 \tag{55}$$

We may split the integral in 0, such that by the properties of limits (and using the definitions of α,β), it is enough to prove separately

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^0 f(x-y)g_\varepsilon(y)\,dy - \int_{-\infty}^0 g(y)f(x_+)\,dy\right] \overset{?}{=} 0 \tag{56}$$

$$\lim_{\varepsilon\to 0}\left[\int_0^\infty f(x-y)g_\varepsilon(y) - \int_0^\infty g(y)f(x_-)\,dy\right] \overset{?}{=} 0 \tag{57}$$

We give here the proof for the "right" side — the "left" side is entirely analogous.

For this to be true, a small enough ε would allow us to make the expression as small as demanded. That is, given $\delta > 0$, for sufficiently small $\varepsilon > 0$

$$\left|\int_{-\infty}^0 f(x-y)g_\varepsilon(y)\,dy - \int_{-\infty}^0 g(z)f(x_+)\,dz\right| \le \delta \tag{58}$$

We change variables in the right integral $z := y/\varepsilon$ and combine them:

$$= \left|\int_{-\infty}^0 f(x-y)g_\varepsilon(y)\,dy - \int_{-\infty}^0 g(y/\varepsilon)f(x_+)\,\frac{dy}{\varepsilon}\right| = \left|\int_{-\infty}^0 g_\varepsilon(y)(f(x-y) - f(x_+))\,dy\right| \tag{59}$$

Note that $y < 0 \implies x < x - y$, so $f(x-y) \to f(x_+)$ as $y \to 0^-$. We therefore split the integral close to zero at some $y_0 < 0$ to examine this behaviour

$$= \left|\int_{-\infty}^{y_0} g_\varepsilon(y)(f(x-y) - f(x_+))\,dy + \int_{y_0}^0 g_\varepsilon(y)(f(x-y) - f(x_+))\,dy\right| \tag{60}$$

$$\le \left|\int_{-\infty}^{y_0} g_\varepsilon(y)(f(x-y) - f(x_+))\,dy\right| + \left|\int_{y_0}^0 g_\varepsilon(y)(f(x-y) - f(x_+))\,dy\right| \tag{61}$$

and estimate each term separately.

Firstly, the integral on $(y_0,0)$:

$$\left|\int_{y_0}^{0} g_\varepsilon(y)(f(x-y)-f(x_+))\,dy\right| \leq \sup_{y\in(y_0,0)} |f(x-y)-f(x_+)| \left|\int_{y_0}^{0} g_\varepsilon(y)\,dy\right| \tag{62}$$

$$\leq \sup_{y\in(y_0,0)} |f(x-y)-f(x_+)| \int_{\mathbb{R}} |g_\varepsilon(y)|\,dy = \{z=y/\varepsilon\} \tag{63}$$

$$= \sup_{y\in(y_0,0)} |f(x-y)-f(x_+)| \int_{\mathbb{R}} |g(z)|\,dz = \sup_{y\in(y_0,0)} |f(x-y)-f(x_+)|\, \|g\|_{\mathcal{L}^1} \leq \frac{\delta}{2} \tag{64}$$

since, by definition of $f(x_+)$, we can choose $y_0 < 0$ so that the supremum becomes arbitrarily small, precisely $\sup_{y\in(y_0,0)} |f(x-y)-f(x_+)| \leq \frac{\delta}{2\|g\|_{\mathcal{L}^1}}$.

Secondly, the remaining integral, for which we have two cases in the theorem: Either (1) f is bounded, i.e. $\exists M>0 : |f(x-y)| \leq M\ \forall x,y$ and $|f(x_+)| \leq M$.

$$\left|\int_{-\infty}^{y_0} g_\varepsilon(y)(f(x-y)-f(x_+))\,dy\right| \leq 2M \int_{-\infty}^{y_0} |g_\varepsilon(y)|\,dy = \{z=y/\varepsilon\} \tag{65}$$

$$= 2M \int_{-\infty}^{y_0/\varepsilon} |g(z)|\,dz \leq \frac{\delta}{2} \tag{66}$$

since, as $\varepsilon \to 0^+$, $y_0/\varepsilon \to -\infty$, and the tail of the convergent integral tends to zero as well. It is sufficient that ε is small enough that the tail is $\leq \delta/4M$; or (2) without loss of generality, g is zero outside $[-R,R]$.

$$\left|\int_{-\infty}^{y_0} g_\varepsilon(y)(f(x-y)-f(x_+))\,dy\right| = \left|\int_{-\infty}^{y_0/\varepsilon} g(z)(f(x-\varepsilon z)-f(x_+))\,dz\right| \tag{67}$$

For $y_0/\varepsilon < -R$, the integrand is zero. It is sufficient that $\varepsilon < -y_0/R$ since obviously $0 \leq \delta/2$ and the integral is bounded by the desired constant.

The sum of these two estimates give the desired estimate for the expression which completes the proof. □

Theorem* (Sampling theorem). *Let $f \in \mathcal{L}^2(\mathbb{R})$. Given that $\exists L > 0$: $\widehat{f}(\xi) = 0\ \forall |\xi| > L$, then*

$$f(t) = \sum_{n\in\mathbb{Z}} f(\frac{n\pi}{L}) \frac{\sin(n\pi - tL)}{n\pi - tL} \tag{68}$$

Proof. Since $\widehat{f}$ has compact support, we may expand it in a Fourier series $\widehat{f}(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\pi x/L}$ where

$$c_n = \frac{1}{2L}\int_{-L}^{L} \widehat{f}(x) e^{-in\pi x/L}\, dx = \frac{1}{2L}\int_{\mathbb{R}} \widehat{f}(x) e^{ix\frac{-n\pi}{L}}\, dx \tag{69}$$

$$\overset{\text{(FIT)}}{=} \frac{2\pi}{2L} f(-\frac{n\pi}{L}) = \frac{\pi}{L} f(-\frac{n\pi}{L}) \tag{70}$$

We now insert this expression into the Fourier transform (noting that the exchange of limits is valid):

$$f(t) \overset{\text{(FIT)}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(x) e^{ixt}\, dx = \frac{1}{2\pi}\int_{-L}^{L} \widehat{f}(x) e^{ixt}\, dx \tag{71}$$

$$= \frac{1}{2\pi}\int_{-L}^{L} \sum_{n\in\mathbb{Z}} c_n e^{in\pi x/L} e^{ixt}\, dx = \frac{1}{2\pi}\int_{-L}^{L} \sum_{n\in\mathbb{Z}} \frac{\pi}{L} f(-\frac{n\pi}{L}) e^{in\pi x/L} e^{ixt}\, dx \tag{72}$$

$$= \frac{1}{2L}\sum_{n\in\mathbb{Z}} f(-\frac{n\pi}{L}) \int_{-L}^{L} e^{i(n\pi/L+t)x}\, dx = \frac{1}{2L}\sum_{n\in\mathbb{Z}} f(-\frac{n\pi}{L}) \left[\frac{e^{i(n\pi/L+t)x}}{i(n\pi/L+t)}\right]_{x=-L}^{L} \tag{73}$$

$$= \frac{1}{2L}\sum_{n\in\mathbb{Z}} f(-\frac{n\pi}{L}) \frac{e^{i(n\pi+tL)} - e^{-i(n\pi+tL)}}{i(n\pi/L+t)} = \sum_{n\in\mathbb{Z}} f(-\frac{n\pi}{L}) \frac{\sin(n\pi+tL)}{n\pi+tL} \tag{74}$$

$$= \{m = -n\} = \sum_{m\in\mathbb{Z}} f(\frac{m\pi}{L}) \frac{\sin(m\pi - tL)}{m\pi - tL} \tag{75}$$

where the substitution in the sum flips the order using the parity (odd) of sine. ☐

Definition (Laplace transform). Assume that $\forall t < 0 : f(t) = 0$ and $\exists a, C > 0$: $|f(t)| \le Ce^{at}$.

$$\mathcal{L}f(s) = \widetilde{f}(s) := \int_0^\infty f(t) e^{-st}\, dt \tag{76}$$

TODO: Laplace properties

Definition (Bessel J i.e. fcn of the first kind)**.** Bessel functions are the solutions to $x^2 f'' + xf' + (x^2 - \nu^2)f = 0$ for $\nu \in \mathbb{C}$. They may be expressed as

$$J_\nu(x) := \sum_{k\geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)} \tag{77}$$

TODO: Bessel functions definitions and important properties

Theorem* (Generating fcn for J_n)**.** *For all x and $z \neq 0$ the Bessel functions of the first kind satisfy*

$$\sum_{n\in\mathbb{Z}} J_n(x) z^n = \exp\left(\frac{x}{2}(z - z^{-1})\right) \tag{78}$$

Proof. Split the RHS into two power series:

$$\exp\left(\frac{xz}{2}\right) = \sum_{j\geq 0} \frac{1}{j!}\left(\frac{xz}{2}\right)^j \qquad \exp\left(-\frac{x}{2z}\right) = \sum_{k\geq 0} \frac{1}{k!}\left(-\frac{x}{2z}\right)^k \tag{79}$$

Combine into the product:

$$\exp\left(\frac{x}{2}(z - z^{-1})\right) = \left(\sum_{j\geq 0} \frac{(xz)^j}{2^j j!}\right)\left(\sum_{k\geq 0} \frac{(-1)^k x^k}{2^k z^k k!}\right) \tag{80}$$

$$= \sum_{j,k\geq 0} \frac{(-1)^k x^{j+k} z^{j-k}}{2^{j+k} j! k!} = \{n := j - k \in \mathbb{Z}\} = \sum_{n\in\mathbb{Z}} \sum_{k\geq 0} \frac{(-1)^k x^{n+2k} z^n}{2^{n+2k}\Gamma(n+k+1)k!} \tag{81}$$

$$= \left\{ J_\nu(x) = \sum_{k\geq 0} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)} \right\} = \sum_{n\in\mathbb{Z}} J_n(x) z^n \tag{82}$$

□

Definition (Hermite polynomials)**.**

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \tag{83}$$

Theorem* (Orthogonality of Hermite polynomials)**.** *The Hermite polynomials are orthogonal on* $\mathcal{L}^2_{\exp(-x^2)}(\mathbb{R})$*.*

Proof. Without loss of generality, we may assume $n > m$.

$$\langle H_n, H_m\rangle_{e^{-x^2}} = \int_{\mathbb{R}} H_n(x)\overline{H_m(x)}e^{-x^2}\,dx \tag{84}$$

$$= \int_{-\infty}^{\infty}(-1)^n e^{x^2}e^{-x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)H_m(x)\,dx = (-1)^n\int_{-\infty}^{\infty}\frac{d^n}{dx^n}\left(e^{-x^2}\right)H_m(x)\,dx \tag{85}$$

We integrate by parts, and use the fact that any polynomial multiplied with e^{-x^2} vanishes at infinity:

$$= \cancelto{0}{(-1)^n\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)H_m(x)\Big|_{-\infty}^{\infty}} - (-1)^n\int_{-\infty}^{\infty}\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)H'_m(x)\,dx \tag{86}$$

$$= (-1)^{n+1}\int_{-\infty}^{\infty}\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right)H'_m(x)\,dx \tag{87}$$

We repeat this another $n-1$ times for a total of n degrees taken out, applying similar reasoning for the boundary terms (they vanish). The result is

$$(-1)^{2n}\int_{-\infty}^{\infty}e^{-x^2}\cancelto{0}{H_m^{(n)}(x)}\,dx = 0 \tag{88}$$

since $\deg H_m(x) = m < n$. □

Theorem* (Generating fcn for Hermite polynomials)**.** *For any* $x \in \mathbb{R}$*,* $z \in \mathbb{C}$*, the Hermite polynomials* $H_n(x)$ *satisfy*

$$\sum_{n\geq 0} H_n(x)\frac{z^n}{n!} = e^{2xz-z^2} \tag{89}$$

Proof. By completing the square, $\exp(2xz - z^2) = \exp(-(x-z)^2)\exp(x^2)$. Expand the left factor in a Taylor series about $z = 0$ (recall the definition):

$$e^{-(x-z)^2}e^{x^2} = e^{x^2}\sum_{n\geq 0}\frac{z^n}{n!}\left[\frac{d^n}{dz^n}e^{-(x-z)^2}\right]_{z=0} = \left\{\begin{matrix} u := x-z \\ du = -dz\end{matrix}\right\} \tag{90}$$

$$= e^{x^2}\sum_{n\geq 0}\frac{(-1)^n z^n}{n!}\left[\frac{d^n}{du^n}e^{-u^2}\right]_{u=x} \stackrel{\text{def}}{=} \sum_{n\geq 0}\frac{H_n(x)}{n!}z^n \tag{91}$$

using the chain rule and the definition of $H_n(x)$. □

TODO: Legendre polynomials, Laguerre polynomials