MVE162

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Theorems marked with * are included on the theory list. Additional theorems, definitions and remarks are included to provide suitable background or techniques for difficult problems. This material is subject to change.

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List of Theorems

Remark (Linear autonomous systems of ODE). We consider the IVP

$$\begin{cases} x'(t) = Ax(t), \quad x(t) \in \mathbb{R}^n, t \in \mathbb{R} \\ x(\tau) = \xi \end{cases} \tag{$\star$}$$

where A is a constant $n \times n$-matrix.

Lemma* (Grönwall inequality)**.**

$$\|x(t)\| \leq \|\xi\| \exp(\|A\| (t - \tau)) \tag{1}$$

where x, ξ, A, τ are as in $(\star)$

Proof. Suppose that $(\star)$ has some solution $x(t)$ on an interval I such that $\tau \in I$. Consider the case when $\tau \leq t$. The equivalent integral equation becomes

$$x(t) = \xi + \int_\tau^t Ax(\sigma)\, d\sigma \tag{2}$$

Taking the norm, applying the triangle inequality twice and using the definition of matrix norm yields

$$\|x(t)\| \overset{\Delta}{\leq} \|\xi\| + \left\| \int_\tau^t Ax(\sigma)\, d\sigma \right\| \overset{\Delta \int}{\leq} \|\xi\| + \int_\tau^t \|Ax(\sigma)\|\, d\sigma \tag{3}$$

$$\leq \|\xi\| + \int_\tau^t \|A\| \, \|x(\sigma)\|\, d\sigma =: G(t) \tag{4}$$

We conclude that the RHS (defined as $G(t)$) satisfies $G(\tau) = \|\xi\|$ and by the Fundamental theorem of Calculus

$$G'(t) = \|A\| \, \|x(t)\| \leq \|A\| \, G(t) \tag{5}$$

Using integrating factor on the inequality we obtain

$$\frac{d}{dt} \left[G(t) \exp(-\|A\| \, t) \right] \leq 0 \tag{6}$$

By integrating both sides over (τ,t) and reordering one obtains the inequality

$$G(t) \leq \|\xi\| \exp(\|A\| (t - \tau)) \tag{7}$$

and we are done, since $\|x(t)\| \leq G(t)$ as seen above. □

Theorem* (Uniqueness of IVP solutions, linear system)**.** *The solution to* $(\star)$ *is unique.*

Proof. Suppose that we have two distinct solutions $x(t), y(t)$ such that $x(\tau) = \xi = y(\tau)$ for $\tau \leq t$. Then, by linearity, $z(t) := x(t) - y(t)$ is a solution, with $z(\tau) = 0$. By Grönwall's inequality and properties of norms

$$\|z(t)\| \leq 0 \implies \|x(t) - y(t)\| = 0 \implies x(t) \equiv y(t) \tag{8}$$

and the solution is unique by contradiction. □

Proposition* (Dimension of solution space, linear system)**.** *Let* $b_1, \ldots, b_N$ *be a basis in* $\mathbb{C}^N$*. Then the functions* $y_j : \mathbb{R} \to \mathbb{C}^N$ *defined as solutions to* $(\star)$ *with* $y_j(\tau) = b_j$*,* $j = 1, \ldots, N$*, that is*

$$y_j(t) = \exp(A(t - \tau))b_j \tag{9}$$

form a basis for the solution space $\mathcal{S}_{hom}$*, and* $\dim \mathcal{S}_{hom} = N$*.*

Proof. Consider a linear combination of $y_j(t)$ equal to zero for some time $\sigma \in \mathbb{R}$.

$$l(\sigma) := \sum_{j=1}^{N} \alpha_j y_j(\sigma) = 0 \tag{10}$$

Observe that the trivial (constant zero) solution coincides with l at this time. By uniqueness, $l(t)$ at arbitrary time t must then coincide with the zero solution $\forall t$ and in particular $t = \tau$. Therefore

$$l(\tau) = \sum_{j=1}^{N} \alpha_j y_j(\tau) = \sum_{j=1}^{N} \alpha_j b_j = 0 \tag{11}$$

and necessarily $\alpha_j = 0 \ \forall j$ since b_j form a basis, which by definition implies $y_1(t), \ldots, y_N(t)$ are linearly independent $\forall t \in \mathbb{R}$.

Arbitrary initial data $x(\tau) = \xi$ can be represented in the basis as $\xi = \sum_{j=1}^{N} C_j b_j$ and the construction above shows that arbitrary solutions can be represented as linear combinations of $y_j(t)$:

$$x(t) = e^{A(t-\tau)}\xi = e^{A(t-\tau)} \sum_{j=1}^{N} C_j b_j = \sum_{j=1}^{N} C_j e^{A(t-\tau)} b_j = \sum_{j=1}^{N} C_j y_j(t) \tag{12}$$

Thus, $\{y_1(t), \ldots, y_N(t)\}$ is a basis for $\mathcal{S}_{\text{hom}}$ and accordingly $\dim \mathcal{S}_{\text{hom}} = N$. $\square$

Corollary* (Sufficient conditions for stability, linear autonomous system)**.** *Let* $A \in \mathbb{C}^{N\times N}$*,* $\mu_A = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ *where* $\sigma(A)$ *is the set of all eigenvalues to* A*. Thus* μ_A *denotes the maximal real part of the eigenvalues to* A*. Then the following statements are valid:*

1. $\|\exp(At)\|$ *decays exponentially iff* $\mu_A < 0$*.*
   ($\exists M_\beta > 0, \beta > 0$ *s.t.* $\|\exp(At)\| \leq M_\beta e^{-\beta t}$*)*
2. $\lim_{t\to\infty} \|\exp(At)\xi\| = 0$ *for every* $\xi \in \mathbb{C}^n$ *iff* $\mu_A < 0$*.*
   (all solutions to $x' = Ax$ *tend to zero)*
3. *if* $\mu_A = 0$ *then* $\sup_{t\geq 0} \|\exp(At)\| < \infty$ *iff all purely imaginary and zero eigenvalues are semisimple (i.e. no generalized eigenvectors or alg. mult. is geom. mult.)*

Proof. Note that any $A \in \mathbb{C}^{N\times N}$ can be represented as $A = TJT^{-1}$ where J is in Jordan canonical form and T is invertible. Furthermore $\|\exp(At)\| = \left\|T\exp(Jt)T^{-1}\right\| \leq \|T\| \left\|T^{-1}\right\| \|\exp(Jt)\|$.

Matrices form a finite dimensional linear space and all norms are equivalent; any two norms $\|\cdot\|_1, \|\cdot\|_2$ such that $\exists C_1, C_2 > 0\, \forall A : C_1 \|A\|_1 \leq \|A\|_2 \leq C_2 \|A\|_1$.

We use the norm $\|A\|_{\max} = \max_{i,j} |A_{ij}|$ (maximum element). Thus, to show boundedness of $\|\exp(Jt)\|$ it is sufficient to show boundedness for all elements in $\exp(Jt)$ (and similarly for the behaviour at infinity).

All elements in $\exp(Jt)$ have one of the forms $\exp(\lambda_i t)$ or $C\exp(\lambda_i t)t^p$ with some $C, p > 0$ depending on block (λ_i may repeat in different blocks). The absolute values then have the form $\exp(\operatorname{Re}\lambda_i \cdot t)$ or $C\exp(\operatorname{Re}\lambda_i \cdot t)t^p$ with $\operatorname{Re}\lambda_i \leq \mu_A$ since $\|\exp(i\operatorname{Im}\lambda_i \cdot t)\| = 1$.

Sufficiency (1): If $\mu_A < 0$ then the maximum of the absolute values satisfies

$$\max_{i,j} |[\exp(Jt)]_{ij}| \leq M\exp((\mu_A + \delta)t) \xrightarrow[t\to\infty]{} 0 \tag{13}$$

tending to zero exponentially for some $M > 0$ and δ such that $-\beta = \mu_A + \delta < 0$:

$$\exp(\operatorname{Re}\lambda_i \cdot t)t^p \leq \exp(\mu_A t)t^p = \exp((\mu_A + \delta - \delta)t)t^p \tag{14}$$

$$= \exp((\mu_A + \delta)t)\underbrace{t^p\exp(-\delta t)}_{\to 0 \implies \leq M} \leq M\exp((\mu_A + \delta)t) = Me^{-\beta t} \tag{15}$$

Sufficiency (2): Definition of matrix norm implies that if $\mu_A < 0$ then

$$\lim_{t\to\infty} \|\exp(At)\xi\| \leq \|\xi\| \lim_{t\to\infty} \|\exp(At)\| = 0 \tag{16}$$

Sufficiency/Necessity (3): If $\mu_A = 0$ and there are purely imaginary or zero eigenvalues λ, then elements in the blocks of $\exp(Jt)$ will have the form 1 or Ct^p by previous reasoning. Therefore the absolute values of these elements will be bounded iff no elements with powers of t are present, i.e. the eigenvalues are semisimple.

Proof of other necessities: see lecture notes (not necessary to learn). □

Remark (Inhomogeneous autonomous systems of ODE). We consider the IVP

$$\begin{cases} x'(t) = Ax(t) + g(t), \quad x(t) \in \mathbb{R}^n, t \in \mathbb{R} \\ x(\tau) = \xi \end{cases} \tag{$I\star$}$$

where A is a constant $n \times n$-matrix and $g : \mathbb{R} \to \mathbb{R}^n$ is (piecewise) continuous.

Proposition* (Duhamel's formula, variant)**.** *The unique solution to* $(I\star)$ *with* $\tau = 0$ *is*

$$x(t) = e^{At}\xi + \int_0^t e^{A(t-\sigma)}g(\sigma)\,d\sigma$$

Proof.

$$x(t) = e^{At}\left(\xi + \int_0^t e^{-A\sigma}g(\sigma)\,d\sigma\right) \tag{17}$$

$$\implies x'(t) = Ae^{At}\left(\xi + \int_0^t e^{-A\sigma}g(\sigma)\,d\sigma\right) + e^{At}e^{-At}g(t) = Ax(t) + g(t) \tag{18}$$

for all points where g is continuous.

Now the difference $z(t) := x(t) - y(t)$ between two solutions satisfies $z'(t) = Az(t)$ and $z(0) = 0$. Uniqueness for homogeneous systems implies $z \equiv 0$ and the solution $x(t)$ is therefore unique. □

Theorem* (Stability of equilibrium points to linear systems perturbed by a small RHS)**.** *Let $G \subset \mathbb{R}^N$ be a non-empty open subset with $0 \in G$. Consider*

$$\begin{cases} x'(t) = Ax(t) + h(x) \\ x(0) = \xi \end{cases} \tag{$P\star$}$$

where $A \in \mathbb{R}^{N\times N}$ and $h : G \to \mathbb{R}^N$ is a continuous function satisfying

$$\lim_{z\to 0} \frac{h(z)}{\|z\|} = 0. \tag{19}$$

If A is Hurwitz ($\operatorname{Re}\lambda < 0\ \forall\lambda \in \sigma(A)$) then 0 is an asymptotically stable equilibrium of $(P\star)$.

Moreover $\exists\Delta > 0, C > 0, \alpha > 0\ \forall \|\xi\| < \Delta : \|x(t)\| \leq C\|\xi\| e^{-\alpha t}$ (for all solutions $x(t)$ to $(P\star)$).

Proof. If $\operatorname{Re}\lambda < 0\ \forall\lambda \in \sigma(A)$ then $\exists\beta > 0 : \operatorname{Re}\lambda < -\beta$ and $\|\exp(At)\| \leq Ce^{-\beta t}$ for some $C > 0$. We can choose $\varepsilon > 0$ such that $C\varepsilon < \beta$ and using (19) choose δ_ε such that for $\|z\| < \delta_\varepsilon$, $z \in G$ it holds that $\|h(z)\| < \varepsilon\|z\|$.

By Picard-Lindelöf we may now conclude that the solution to $(P\star)$ exists on some interval $t \in [0, \delta)$ and we may apply the Duhamel formula:

$$x(t) = \exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))\,d\sigma \tag{20}$$

As long as $x(\sigma)$ is such that $\{x : \|x\| \leq \delta_\varepsilon\} \subset G$ the triangle inequality for integrals applies as follows:

$$\|x(t)\| \leq \|\exp(At)\|\,\|\xi\| + \int_0^t \|\exp(A(t-\sigma))\|\,\|h(x(\sigma))\|\,d\sigma \tag{21}$$

$$\leq Ce^{-\beta t}\|\xi\| + \int_0^t Ce^{-\beta(t-\sigma)}\varepsilon\|x(\sigma)\|\,d\sigma \tag{22}$$

Let $y(t) := \|x(t)\| e^{\beta t}$. Multplying by $e^{\beta t}$ yields

$$y(t) \leq C\|\xi\| + \int_0^t (C\varepsilon)y(\sigma)\,d\sigma \tag{23}$$

and the Grönwall inequality implies

$$\|y(t)\| \leq C\|\xi\| e^{C\varepsilon t} \implies \|x(t)\| \leq C\|\xi\| e^{-(\beta - C\varepsilon)t} \tag{24}$$

as long as $\|x(t)\| \leq \delta_\varepsilon$. Let $\alpha = \beta - C\varepsilon > 0$ (requiring ε small enough), $\Delta = \frac{\delta_\varepsilon}{2C}$ and $\|\xi\| < \Delta$. Such a choice implies $\|\xi\| \leq \delta_\varepsilon$, if this solution exists.

Important argument. The last estimate in fact implies that the solution must exist on $\mathbb{R}_+$. Suppose that there exists a maximal existence time $t_{\max}$. Firstly, using continuity and boundedness of $x(t)$ on $[0, t_{\max})$ with the integral form the orbit set $\{x(t) : t \in [0, t_{\max})\}$ is bounded by $\|\xi\| \leq \delta_\varepsilon$. The closure $\mathcal{C}$ is therefore compact and thus $h(x)$ (continuous on G) is bounded on $\mathcal{C}$. Therefore the following limit exists:

$$\lim_{t\to t_{\max}} \int_0^t e^{A(t-\sigma)}h(x(\sigma))\,d\sigma. \tag{25}$$

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to t_{\max}$ the sequence of integrals

$$\{I_k = \int_0^{t_k} e^{A(t_k-\sigma)} h(x(\sigma))\, d\sigma\}_{k=1}^{\infty} \tag{26}$$

is a Cauchy sequence because

$$\|I_m - I_k\| \leq \left\| \int_{t_k}^{t_m} e^{A(t_m-t_k-\sigma)} h(x(\sigma))\, d\sigma \right\| \leq C|t_m - t_k| \to 0,\ m,k \to \infty \tag{27}$$

Thus we may extend

$$x(t_{\max}) = \lim_{t\to t_{\max}} x(t) = \lim_{t\to t_{\max}} \left(\exp(At)\xi + \int_0^t \exp(A(t-\sigma))h(x(\sigma))\, d\sigma \right) =: \eta. \tag{28}$$

Secondly, using the existence theorem, there is a solution to $y'(t) = Ay + h(y)$ on $[t_{\max}, t_{\max}+\delta)$ with IC $y(t_{\max}) = \eta$. This is evidently an extension of $x(t)$ which contradicts the assumption of a maximal existence time, and thus $x(t)$ may in fact be extended to $\mathbb{R}_+$ still satisfying our desired estimate and the asymptotic stability in the origin follows. □

Corollary (Chapman-Kolmogorov). *For all $t,\sigma,\tau \in J$ the transition matrix function $\Phi(t,\tau)$ satisfies*

$$\Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau) \tag{29}$$

$$\Phi(\tau,\tau) = I \tag{30}$$

$$\Phi(\tau,t)\Phi(t,\tau) = \Phi(\tau,\tau) = I \tag{31}$$

$$\Phi(\tau,t) = (\Phi(t,\tau))^{-1} \tag{32}$$

Theorem* (Floquet representation). *Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of the monodromy matrix $\Phi(p,0)$. There exists a periodic (with period p) piecewise continuously differentiable function $\Theta : \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular (invertible, non-zero eigenvalues) for all t, such that*

$$\Phi(t,0) = \Theta(t) \exp\left(\frac{t}{p}G\right) \tag{33}$$

Proof. Recall the main property of the monodromy matrix; for $\tau = 0$

$$\Phi(t+p,0) \overset{\text{CK}}{=} \Phi(t+p,p)\Phi(p,0) \overset{\text{S}}{=} \Phi(t,0)\Phi(p,0) \tag{34}$$

We denote $\frac{1}{p}G =: F$ for convenience, so that $\log(\Phi(p,0)) = G = pF$ and let

$$\Theta(t) := \Phi(t,0)\exp\left(-\frac{t}{p}G\right) = \Phi(t,0)\exp(-tF) \tag{35}$$

which is well-defined. We now show it has the desired properties: (1) periodicity p and (2) satisfies the initial condition.

Recall $\Theta(0) = I$ and $\Theta(np) = I$, $n = 0,1,2\ldots$. It holds that $\Phi(t,0)$ is (pw) continuous if $A(t)$ is (pw) continuous. Therefore $\Theta(t)$ is (pw) continuous, because $\exp(-tF)$ is continuously differentiable. Also, $\Theta(t)$ is invertible for all t as a product of two non-singular matrices.

We now check

$$\Theta(t+p,0) = \Phi(t+p,0)\exp(-(t+p)F) = \Phi(t+p,0)\exp(-G)\exp(-tF) \tag{36}$$
$$= \left\{e^{-G} = (e^{G})^{-1} = (\Phi(p,0))^{-1} = \Phi(0,p)\right\} = \Phi(t+p,0)\Phi(0,p)\exp(-tF) \tag{37}$$
$$\overset{\text{M}}{=} \Phi(t,0)\Phi(p,0)\Phi(0,p)\exp(-tF) = \Phi(t,0)\exp(-tF) = \Theta(t). \tag{38}$$

□

Theorem* (Floquet multiplier boundedness)**.** *The Floquet multipliers are the eigenvalues of the monodromy matrix. Every solution to a periodic linear system*

(i) is bounded on $\mathbb{R}_+$ iff the absolute value of each Floquet multiplier ≤ 1 and any Floquet multiplier with absolute value 1 is semisimple.

(ii) tends to zero as $t \to \infty$ iff the absolute value of each Floquet multiplier is < 1.

Proof. By the Floquet theorem any solution $x(t)$ to $x'(t) = A(t)x(t)$, $A(t+p) = A(t)$ $\forall t \in \mathbb{R}$ satisfying $x(\tau) = \xi$ is represented as

$$x(t) = \Theta(t)\exp(tF)\Phi(0,\tau)\xi = \Theta(t)\exp(tF)\zeta \tag{39}$$

where $F = \frac{1}{p}\operatorname{Log}(\Phi(p,0))$, $\zeta = \Phi(0,\tau)\xi \in \mathbb{R}^N$, and $\Theta(t)$ is a p-periodic invertible (pw) continuous matrix function.

We define $y(t) = \exp(tF)\zeta$ as a solution to $y' = Fy$, $y(0) = \zeta$.

$$y(t) = \Theta^{-1}(t)x(t) \iff x(t) = \Theta(t)y(t) \tag{40}$$

The mapping Θ defines a one-to-one correspondence between x and y. Periodicity and continuity of $\Theta(t)$ imply that $\exists M > 0$:

$$\|\Theta(t)\|, \left\|\Theta^{-1}(t)\right\| \leq M \ \forall t \in \mathbb{R} \implies \|x(t)\| \leq M\,\|y(t)\|,\ \|y(t)\| \leq M\,\|x(t)\| \tag{41}$$

Therefore

(1) $\|x(t)\|$ is bounded on $\mathbb{R}_+$ iff $\|y(t)\| = \left\|e^{tF}\zeta\right\|$ is bounded on $\mathbb{R}_+$

(2) $\|x(t)\| \to 0$ as $t \to \infty$ iff $\|y(t)\| \to 0$ as $t \to \infty$

Since $\log(\Phi(p,0)) = G = pF$, it follows that

$$\sigma(\Phi(p,0)) = \{\exp(\lambda p) : \lambda \in \sigma(F)\} \tag{42}$$
$$\sigma(F) = \{\tfrac{1}{p}\operatorname{Log}(\mu) : \mu \in \sigma(\Phi(p,0))\} \tag{43}$$

when algebraic and geometric multiplicities coincide. Recall $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg} z$, $\exp(z) = \exp(\operatorname{Re} z)(\cos(\operatorname{Im} z) + i\sin(\operatorname{Im} z))$. The Floquet multiplier μ (and the corresponding eigenvalue λ to F) has

(a) $|\mu| < 1$ iff $\operatorname{Re}\lambda < 0$

(b) $|\mu| \leq 1$ iff $\operatorname{Re}\lambda \leq 0$

(c) $|\mu| = 1$ and semisimple iff $\operatorname{Re}\lambda = 0$ and semisimple.

Known relations between solutions to an autonomous system and the spectrum of corresponding matrix imply properties of $y(t)$ and thus with 1) 2) a) b) c) imply the theorem. □

Remark (Non-linear systems of ODE). We consider the IVP

$$\begin{cases} x'(t) = f(t,x), \quad f : J \times G \to \mathbb{R}^n \\ x(\tau) = \xi \end{cases} \tag{$\star\star$}$$

where $J \subset \mathbb{R}$ is an interval, $G \subset \mathbb{R}^n$ open, $(\tau, \xi) \in J \times G$, f continuous in $J \times G$.

Corollary ('Eternal life' of solutions enclosed in compact set)**.** *Let $x : I_\xi \to G$ be a maximal solution to ($\star\star$). Suppose that its positive semi-orbit $O_+(\xi)$ is contained in a compact subset $C \subset G$. Then I_ξ is infinite to the right (with respect to J), i.e. $J \cap [\tau,\infty) \subseteq I_\xi$. Similar statements apply for the negative semi-orbit $O_-(\xi)$ and backwards time, and the orbit $O(\xi)$ and the whole J.*

Proposition* (Extensibility of solutions, linear bound on RHS)**.** *Consider the IVP ($\star\star$), with f locally Lipschitz in x. Assume that for any compact interval $K \subset J$ there exists $L > 0$ such that for $t \in K$ the RHS does not grow faster than linearly:*

$$\|f(t,x)\| \leq L(1 + \|x\|). \tag{44}$$

If $x : I \to \mathbb{R}^N$ is a maximal solution to the equation, then $I = J$.

Proof. Define $\omega := \sup I$, $\alpha := \inf I$. Suppose the statement is false; e.g. $\omega \in J$, $\omega \notin I$, and $\tau < \omega$. Choose the constant L such that the estimate above is valid for $t \in [\tau,\omega]$. Using the integral form and the triangle inequality for integrals,

$$\begin{aligned} \|x(t)\| &\leq \|x(\tau)\| + \int_\tau^t \|f(s,x(s))\| \, ds \leq \|x(\tau)\| + L \int_\tau^t (1 + \|x(s)\|) \, ds \\ &= \|x(\tau)\| + L(t-\tau) + L \int_\tau^t \|x(s)\| \, ds \quad \forall t \in [t,\omega) \end{aligned} \tag{45}$$

By Grönwall's inequality, $\|x(t)\|$ is bounded by some constant C on $[t,\omega)$. Thus the corresponding orbit $\{x(t) : t \in [t,\omega)\}$ is bounded by a compact. Lemma 4.9 implies that the solution can be extended to the closed interval $[t,\omega]$, and actually by existence theorem to an even larger interval beyond, which contradicts the supposition that I is a maximal interval.

The proof is analogous for $\alpha \in J$, $\alpha \notin I$, and $\tau > \alpha$. □

Proposition (Properties of ω-limit sets)**.** *Let $\xi \in G$. Let the closure of the positive semi-orbit $\overline{O^+(\xi)}$ be compact and contained in G. Then $\mathbb{R}_+ \subset I_\xi$ and the ω-limit set $\Omega(\xi) \subset G$ is (1) non-empty, (2) compact, (3) connected, (4) invariant (both positively and negatively) under the local flow, and (5) trajectories approach $\Omega(\xi)$ as $t \to \infty$.*

Theorem (Poincaré-Bendixson)**.** *Suppose that $\xi \in G \subset \mathbb{R}^2$ is such that the closure of the positive orbit $O_+(\xi)$ is compact and contained in G, and the ω-limit set $\Omega(\xi)$ does not contain any equilibrium points. Then $\Omega(\xi)$ is an orbit of a periodic solution.*

Proposition* (Bendixson's criterion for non-existence of periodic solutions)**.** *Let $x' = f(x) = [f_1(x), f_2(x)]^\mathsf{T}$ with $f : G \to \mathbb{R}^2$, $G \subset \mathbb{R}^2$ open, $f \in \mathcal{C}^1(G)$ (although in pratice locally Lipschitz suffices), and let $D \subset G$ be a simply connected domain.*

Suppose that $\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is strictly positive (or strictly negative) in D. Then the equation has no periodic solutions with orbits inside D.

Proof. Suppose it were not so; that there is a periodic trajectory $x(t)$ with period $T > 0$ in D and $x(t+T) = x(t)$. Denote $x_1'(t) = f_1(x(t))$, $x_2'(t) = f_2(x(t))$, and the orbit of $x(t)$ by $\mathcal{L} = \{x(t) : t \in [0,T]\}$. It will be a closed simple curve. Denote the interior domain by Ω. Since $D \supset \Omega$ is simply connected $\partial\Omega = \mathcal{L}$. By Gauss theorem:

$$I := \iint_\Omega \nabla \cdot f \, dx_1 dx_2 = \int_{\partial\Omega} f \cdot n \, dl \tag{46}$$

where n is the outward normal to the boundary $\partial\Omega$. Note $f(x(t)) = x'(t)$ on $\partial\Omega = \mathcal{L}$ because it is an orbit. Therefore $f(x(t))$ is the tangent vector to $\partial\Omega$, $f \perp n$, and $I = 0$. On the other hand $\nabla \cdot f > 0$ (or < 0) in the whole $D \supset \Omega$. Therefore the integral I over a bounded domain Ω must be strictly positive (or negative), which is a contradiction, and the system cannot have a periodic orbit in D. □

Theorem* (Stability by Lyapunov function)**.** *Consider the system $x' = f(x)$, $f : G \to \mathbb{R}^N$ locally Lipschitz continuous, $G \subset \mathbb{R}^N$ open. Let $0 \in G$ be an equilibrium point. Suppose there exists $V : U \to \mathbb{R}$ positive definite continuously differentiable ($\mathcal{C}^1(U)$) such that $U \subset G$, $0 \in U$ and*

$$\frac{dV}{dt} = V_f(z) = \nabla V \cdot f(z) \leq 0 \quad \forall z \in U \tag{47}$$

Then 0 is a stable equilibrium point.

Proof. Take arbitrary $\varepsilon > 0$ such that $B(\varepsilon,0) \subset U$. Let $\alpha := \min_{z \in \partial B(\varepsilon,0)} V(z)$, which exists since the sphere $\partial B(\varepsilon,0)$ is compact and V continuous. Then

$\alpha > 0$ because $V(z) > 0$ outside the equilibrium point. By continuity of V and $V(0) = 0$ one can find $\delta \in (0,\varepsilon)$ such that $\forall \in B(\delta,0)$ it holds that $V(z) < \alpha/2$.

On the other hand, for any part of the trajectory $x(t) = \varphi(t,\xi)$ inside U, the function $V(\varphi(t,\xi))$ is non-increasing because $\dot{V}(\varphi(t,\xi)) \leq 0$. Therefore all trajectories $\varphi(t,\xi)$ with initial condition $\xi \in B(\delta,0)$ satisfy $V(\xi) < \alpha/2$, implying $V(\varphi(t,\xi)) < \alpha/2$ and $\varphi(t,\xi)$ cannot reach $\partial B(\varepsilon,0)$ where $V(z) \geq \alpha$. Any such trajectory therefore stays within $B(\varepsilon,0)$ and the equilibrium point is stable by definition. Also, $\mathbb{R}^+ \subset I_\xi$, since the trajectory stays inside a compact set. □

Theorem* (LaSalle's invariance principle). *Suppose $f : G \to \mathbb{R}^n$ is locally Lipschitz and let $\varphi(t,\xi)$ denote the flow generated by the system $x' = f(x)$. Let $U \subset G$ be non-empty and open. Let $V : U \to \mathbb{R}$ continuously differentiable such that $V_f(z) \leq 0$ for all $z \in U$. If $\xi \in U$ is such that the closure of the positive semi-orbit $O^+(\xi)$ is compact and contained in U, then*

(i) $\mathbb{R}_+ \subset I_\xi$, the maximal interval for IC ξ

(ii) as $t \to \infty$, $\varphi(t,\xi)$ approaches the largest invariant set contained in $V_f^{-1}(0) = \{z \in U : V_f(z) = 0\}$.

Proof. Let $x(t) := \varphi(t,\xi)$. By continuity of V and compactness of the closure $\overline{O^+(\xi)}$, V is bounded on $O^+(\xi)$ and therefore the function $V(x(t))$ is bounded. Since $\frac{d}{dt}V(x(t)) = V_f(x(t)) \leq 0 \ \forall t \in \mathbb{R}_+$, $V(x(t))$ is non-increasing. We conclude that $\lim_{t\to\infty} V(x(t))$ must exist and is finite $=: \lambda$.

Take an arbitrary $z \in \Omega(\xi)$ (the ω-limit set). By definition, there exists a sequence $\{t_k\} \in \mathbb{R}_+$ such that $t_k \to \infty$ as $k \to \infty$ and $x(t_k) \to z$ as $k \to \infty$. By continuity of V,

$$V(z) = \lim_{k\to\infty} V(x(t_k)) = \lim_{t\to\infty} V(x(t)) = \lambda \tag{48}$$

Consequently, $V(z) = \lambda \ \forall z \in \Omega(\xi)$!

By the invariance of $\Omega(\xi)$ with respect to φ, if $z \in \Omega(\xi)$ then $\varphi(t,z) \in \Omega(\xi) \ \forall t \in \mathbb{R}$. Hence, $V(\varphi(t,z)) = \lambda \ \forall t \in \mathbb{R}$, and furthermore

$$V_f(\varphi(t,z)) = \frac{d}{dt}V(\varphi(t,z)) = \frac{d}{dt}\lambda = 0 \ \forall t \in \mathbb{R} \tag{49}$$

Since $\varphi(0,z) = z$ and $z \in \Omega(\xi)$ is arbitrary it follows that $V_f(z) = 0 \ \forall z \in \Omega(\xi)$ and hence $\Omega(\xi) \subset V_f^{-1}(0)$.

The statement now follows from the main theorem about limit sets (4.38) that states $\Omega(\xi)$ is invariant and $x(t)$ approaches $\Omega(\xi)$ as $t \to \infty$. □

Lemma (Banach contraction principle). *Let A be a non-empty closed subset of a Banach space X and $K : A \to A$ be a a contraction operator with contraction constant $\theta < 1$, i.e.*

$$\|K(x) - K(y)\|_X \leq \theta \|x - y\|_X \quad \forall x,y \in A. \tag{50}$$

Then there is a unique fixed point $\overline{x} \in A$ to K such that $K(\overline{x}) = \overline{x}$ and

$$\|K^n(x_0) - \overline{x}\|_X \leq \frac{\theta^n}{1-\theta} \|K(x_0) - \overline{x}\|_X \quad \forall x_0 \in A \tag{51}$$

where $K^n = K \circ \cdots \circ K$ applied n times.

Theorem* (Picard-Lindelöf). *Let $J \subset \mathbb{R}$ be an interval, $G \subset \mathbb{R}^n$ be open, $\tau \in J$, $\xi \in G$, f be continuous in $J \times G$. If f is Lipschitz with respect to $x \in G$ with Lipschitz constant $L > 0$, there is a unique solution $x : I \to \mathbb{R}^n$ to the IVP. (A stronger version uses local Lipschitz conditions, and combines the theorem about maximal extensions.)*

Proof. Consider the operator derived from the integral form of the corresponding IVP:

$$K(x)(t) = \xi + \int_\tau^t f(s,x(s))\, ds \tag{52}$$

on the Banach space of continuous functions $x : I \to \mathbb{R}^n$ on some compact interval $I \subset J$. Let $I = [\tau, \tau + T] \subset J$, for some $T > 0$. (Considering the backwards direction is done in a similar way.)

Firstly, we find conditions on T and a subset $A \subset C(I)$ such that K maps A to itself: $K : A \to A$. Choose first a closed ball $\overline{B(\xi,\delta)} = \{x : \|x - \xi\| \leq \delta\}$ such that it belongs to G. The function $f(t,x)$ is continuous on the compact set $V = I \times \overline{B(\xi,\delta)} \subset \mathbb{R}^{n+1}$ and therefore

$$M := \sup_{(t,x)\in V} \|f(t,x)\| < \infty. \tag{53}$$

Then,

$$\|K(x(t)) - \xi\| = \left\| \int_\tau^t f(s,x(s))\, ds \right\| \leq \int_\tau^t \|f(s,x(s))\|\, ds \leq TM \tag{54}$$

so by choosing $T < \delta/M$ it holds that $\|K(x(t)) - \xi\|\ \ \forall t \in I$. Taking the supremum of both sides yields

$$\sup_{t\in I} \|K(x(t)) - \xi\| = \|K(x) - \xi\|_{C(I)} \leq \delta \tag{55}$$

and hence the operator K maps the closed ball $A \subset C(I)$ defined by the inequality $\|x - \xi\|_{C(I)} \leq \delta$ (when $T < \delta/M$) into itself ($K : A \to A$).

Secondly, we find conditions on T such that K is a contraction on the subset $A \subset C(I)$. Consider the difference between two $x,y \in C(I)$, applying the triangle inequality and the Lipschitz property for the appropriate estimation:

$$\begin{aligned}
\|K(x(t)) - K(y(t))\| &= \left\| \int_\tau^t f(s,x(s)) - f(s,y(s))\, ds \right\| \\
&\overset{\Delta}{\leq} \int_\tau^t \|f(s,x(s)) - f(s,y(s))\|\, ds \overset{\text{L}}{\leq} L \int_\tau^t \|x(s) - y(s)\|\, ds \\
&\overset{\Delta \int}{\leq} LT \sup_{s\in I} \|x(s) - y(s)\| = LT\, \|x - y\|_{C(I)}
\end{aligned} \tag{56}$$

which implies that for $T < 1/L$ the contraction property holds.

Therefore, choosing $T < \min\{\delta/M, 1/L\}$ we conclude that the operator K maps the closed ball $A = \{x \in C(I) : \|x - \xi\|_{C(I)} \leq \delta\}$ into itself ($K : A \to A$) and that K is a contraction on A, i.e. $\|K(x) - K(y)\|_{C(I)} \leq \theta\, \|x - y\|_{C(I)}$, $\theta < 1$, for any $x,y \in A$. By the Banach contraction principle K has for $T < \min\{\delta/M, 1/L\}$ a unique fixed point $\overline{x} \in A$ that is the solution to the IVP. □